A Semantic Approach to Decidability in Epistemic Planning (Supplementary Material)

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Appendix

In what follows, we provide the full proofs of our results. The sections of the Appendix are named after the corresponding (sub)sections of the paper. To enhance clarity, results that are *not* present in the paper are numbered referring to the correspondent Appendix section.

A Epistemic Planning with Commutativity

Lemma 1. *Let $G = \{i_1, \ldots, i_m\} \subseteq \mathcal{AG}$, with $m \geq 2$ and let $\vec{v} \in G^*$ ($|\vec{v}| = \lambda \geq 2$). Let π and ρ be two permutations of elements of $\vec{v}$. Then, for any φ, in the logic C-S5$_n$ the following is a theorem:*

$$\Box_{\pi_1} \ldots \Box_{\pi_\lambda}\varphi \leftrightarrow \Box_{\rho_1} \ldots \Box_{\rho_\lambda}\varphi$$

Proof. First, we notice that in the logic C-S5$_n$, for any formula φ, the following formula is a theorem:

$$\Box_i\Box_j\varphi \leftrightarrow \Box_j\Box_i\varphi \tag{1}$$

This immediately follows from axiom **C**.

Second, by construction, we have that for each π_i there exists ρ_{k_i} such that $\rho_{k_i} = \pi_i$. Consider $\rho_{k_1} = \pi_1$. Then, by iterating Equation 1, we obtain:

$$\begin{aligned}
&\Box_{\rho_1} \ldots \Box_{\rho_{k_1-1}}\Box_{\rho_{k_1}}\Box_{\rho_{k_1+1}} \ldots \Box_{\rho_\lambda}\varphi\\
\leftrightarrow\ &\Box_{\rho_1} \ldots \Box_{\rho_{k_1}}\Box_{\rho_{k_1-1}}\Box_{\rho_{k_1+1}} \ldots \Box_{\rho_\lambda}\varphi\\
\ldots&\\
\leftrightarrow\ &\Box_{\rho_1}\Box_{\rho_{k_1}} \ldots \Box_{\rho_{k_1-1}}\Box_{\rho_{k_1+1}} \ldots \Box_{\rho_\lambda}\varphi\\
\leftrightarrow\ &\Box_{\rho_{k_1}}\Box_{\rho_1} \ldots \Box_{\rho_{k_1-1}}\Box_{\rho_{k_1+1}} \ldots \Box_{\rho_\lambda}\varphi
\end{aligned}$$

By repeating this manipulation for $\pi_2, \ldots \pi_m$, we obtain the conclusion. □

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Lemma 2. *Let $G = \{i_1, \ldots, i_m\} \subseteq \mathcal{AG}$, with $m \geq 2$. In the logic C-S5$_n$, for any φ and $\vec{v} \in G^*$ we have that $\Box_{i_1} \ldots \Box_{i_m}\varphi \rightarrow \Box_{v_1} \cdots \Box_{v_{|\vec{v}|}}\varphi$ is a theorem.*

Proof. The proof is by induction on $|\vec{v}|$. For the base case ($|\vec{v}| = 0$) we have that the formulae $\Box_{i_h}\Box_{i_{h+1}} \ldots \Box_{i_m}\varphi \rightarrow \Box_{i_{h+1}} \ldots \Box_{i_m}\varphi$ ($1 \leq h < m$) and $\Box_{i_m}\varphi \rightarrow \varphi$ are instances of **T**. Together with propositional reasoning, we get that $\Box_{i_1} \ldots \Box_{i_m}\varphi \rightarrow \varphi$ is a theorem.

Let now $|\vec{v}| = \lambda$ and suppose, by inductive hypothesis, that $\Box_{i_1} \ldots \Box_{i_m}\varphi \rightarrow \Box_{v_1} \cdots \Box_{v_\lambda}\varphi$ is a theorem (for any formula φ). We now show that, for each $j \in G$, the formula $\Box_{i_1} \ldots \Box_{i_m}\varphi \rightarrow \Box_{v_1} \cdots \Box_{v_\lambda}\Box_j\varphi$ is also a theorem. By inductive hypothesis, substituting φ with $\Box_j\varphi$, the following is a theorem:

$$\Box_{i_1} \ldots \Box_{i_m}\Box_j\varphi \rightarrow \Box_{v_1} \cdots \Box_{v_\lambda}\Box_j\varphi.$$

Since $j \in G$, there exists $h \in \{1, \ldots m\}$ such that $j = i_h$. From this and Lemma 1, we can rewrite the antecedent of the above implication, $\Box_{i_1} \ldots \Box_{i_m}\Box_j\varphi$, as $\Box_{i_1} \ldots \Box_j\Box_j \ldots \Box_{i_m}\varphi$. Moreover, it is easy to prove that from axioms **T** and **4** the following formula is a theorem (for any formula φ):

$$\Box_j\varphi \leftrightarrow \Box_j\Box_j\varphi. \tag{2}$$

Thus, we can rewrite the above formula as $\Box_{i_1} \ldots \Box_j \ldots \Box_{i_m}\varphi$, which is simply $\Box_{i_1} \ldots \Box_{i_m}\varphi$. Finally, we obtain that the following is a theorem:

$$\Box_{i_1} \ldots \Box_{i_m}\varphi \rightarrow \Box_{v_1} \cdots \Box_{v_\lambda}\Box_j\varphi.$$

This is the required result. □

Lemma 3. *Let (M, W_d) be an C-S5$_n$-state, with $M = (W, R, V)$. For any $w, v \in W$, we have that $w \underline{\leftrightarrow}_{n+1} v \Leftrightarrow w \underline{\leftrightarrow} v$.*

Proof. Clearly, if $w \underline{\leftrightarrow} v$, then $w \underline{\leftrightarrow}_{n+1} v$. For the other direction, assume $w \underline{\leftrightarrow}_{n+1} v$. First recall that there exists a path between any two worlds w and v of length at most n (Corollary 1). We refer to this property as (†). By contradiction, assume that that it is not the case that $w \underline{\leftrightarrow} v$, namely there exist two worlds $w', v' \in W$ such that:

- $w' \underline{\leftrightarrow}_0 v'$;

- w' is reached by a path π_w starting from w (with $|\pi_w|=\ell$);
- v' is reached by a path π_v starting from v (with $|\pi_v|=\ell$);
- There exists a world $w'' \in W$ such that $w' R_{i_{\ell+1}} w''$ (with $i_{\ell+1} \in \mathcal{AG}$), such that for all worlds $v'' \in W$ such that $v' R_{i_{\ell+1}} v''$, it is not the case that $w'' \underline{\leftrightarrow}_0 v''$ and, thus, that $w' \underline{\leftrightarrow}_1 v'$ (or vice-versa, swapping w' with v' and w'' with v'').

Let us denote these two paths as: $\pi_w = wR_{i_1} \circ \cdots \circ R_{i_\ell} w' R_{i_{\ell+1}} w''$ and $\pi_v = vR_{i_1} \circ \cdots \circ R_{i_\ell} v' R_{i_{\ell+1}} v''$, with each $i_x \in \mathcal{AG}$ for $1 \leq x \leq \ell + 1$.

Clearly, $\ell \geq n+1$ otherwise this would contradict the hypothesis that $w \underline{\leftrightarrow}_{n+1} v$. Assume $\ell = n+1$. Thus, $|\pi_w| = |\pi_v| = n+2$. We now show that $w' \underline{\leftrightarrow}_1 v'$. From (†) it follows that there exist two shorter paths $\pi'_w = wR_{j_1} \circ \cdots \circ R_{j_n} w' R_{j_{n+1}} w''$ and $\pi'_v = vR_{j_1} \circ \cdots \circ R_{j_n} v' R_{j_{n+1}} v''$, with each $j_x \in \mathcal{AG}$ for $1 \leq x \leq n+1$.

Since by hypothesis $w \underline{\leftrightarrow}_{n+1} v$, this means that v'' as above exists and also that $w'' \underline{\leftrightarrow}_0 v''$ for any such w'' and v''. This implies $w' \underline{\leftrightarrow}_1 v'$ and thus $w \underline{\leftrightarrow}_{n+2} v$. Since the same argument applies for any $\ell > n+1$, we obtain that $w \underline{\leftrightarrow} v$.

□

Lemma 4. *Let (M, W_d) be a bisimulation-contracted C-S5$_n$-state, with $M = (W, R, V)$. Then, $|W|$ is bounded in n and $|\mathcal{P}|$.*

Proof. Given $k \geq 0$ and a world $w \in W$, we define its k-*characteristic formula* χ_w^k as in [3]:

$$\chi_w^k = \begin{cases} L_w, & \text{if } k = 0 \\ L_w \wedge \bigwedge_{i \in \mathcal{AG}} \left(\text{forth}_{w,i}^k \wedge \text{back}_{w,i}^k\right) & \text{otherwise} \end{cases},$$

where:

$$L_w = \bigwedge_{p | w \in V(p)} p \wedge \bigwedge_{p | w \notin V(p)} \neg p$$
$$\text{forth}_{w,i}^k = \bigwedge_{w' | wR_i w'} \Diamond_i \chi_{w'}^{k-1}$$
$$\text{back}_{w,i}^k = \Box_i \bigvee_{w' | wR_i w'} \chi_{w'}^{k-1}$$

We recall the following well-known result from the literature [3, Theorem 32]:

Claim ($\star$). *The following statements are equivalent*

1. $(M, w) \models \chi_v^k$;
2. $w \underline{\leftrightarrow}_k v$.

By using Claim ($\star$), Lemma 3 and minimality of models w.r.t. bisimulation, we obtain that for any $w, v \in W$, it holds:

$$(M, w) \models \chi_v^{n+1} \Leftrightarrow w \underline{\leftrightarrow}_{n+1} v \Leftrightarrow w = v.$$

Clearly, the size of the set $\{\chi_w^{n+1} \mid w \in W\}$ is bounded in n and $|\mathcal{P}|$, and, hence, the number of worlds of W is also bounded. □

B Generalizing the Principle of Commutativity

B.1 *b*-Commutativity

In this section, we give the full proofs of Theorems 4 and 5 of Section 4.1.

B.1.1 Proof of Theorem 4

To prove Theorem 4, we first show some propaedeutical results (Lemma B.1, Theorem B.1, Corollary B.1 and Lemmata B.2, B.3). Notice that we follow step by step the proof of Theorem 3 and we give the corresponding results in the logic C^b-S5$_2$, for any $b > 1$. Since we consider the specific case involving 2 agents, we fix $\mathcal{AG} = \{0, 1\}$.

The following is the corresponding version of Lemma 2 of Section 4.

Lemma B.1. *Let $i, j \in \mathcal{AG}$ with $i \neq j$, let $\vec{v} \in \mathcal{AG}^*$ and let φ be any formula. Then, for any b>1, in the logic C^b-S5$_2$ the formula $(\Box_i\Box_j)^b\varphi \rightarrow \Box_{v_1} \cdots \Box_{v_{|\vec{v}|}}\varphi$ is a theorem.*

Proof. The proof is by induction on $|\vec{v}|$. For the base case ($|\vec{v}| = 0$) we have that the formulae $(\Box_i\Box_j)^a\varphi \rightarrow \Box_j(\Box_i\Box_j)^{a-1}\varphi$ and $\Box_j(\Box_i\Box_j)^{a-1}\varphi \rightarrow (\Box_i\Box_j)^{a-1}\varphi$ (for each $1 \leq a \leq b$) are instances of **T**. Together with propositional reasoning, we get that $(\Box_i\Box_j)^b\varphi \rightarrow \varphi$ is a theorem.

Let now $|\vec{v}| = \lambda$ and suppose by induction that $(\Box_i\Box_j)^b\varphi \rightarrow \Box_{v_1} \cdots \Box_{v_\lambda}\varphi$ is a theorem (for any formula φ). We now show that, for each $k \in \mathcal{AG}$, the formula $(\Box_i\Box_j)^b\varphi \rightarrow \Box_{v_1} \cdots \Box_{v_\lambda}\Box_k\varphi$ is also a theorem. By inductive hypothesis, substituting φ with $\Box_k\varphi$, the following is a theorem:

$$(\Box_i\Box_j)^b\Box_k\varphi \rightarrow \Box_{v_1} \cdots \Box_{v_\lambda}\Box_k\varphi$$

There are now two cases: either $k = j$, or $k = i$. In the former case, we use Equation 2 as in the proof of Lemma 2 to rewrite the antecedent of the above implication as follows: $(\Box_i\Box_j)^b\Box_j\varphi \equiv (\Box_i\Box_j)^{b-1}\Box_i\Box_j\Box_j\varphi \equiv (\Box_i\Box_j)^{b-1}\Box_i\Box_j\varphi \equiv (\Box_i\Box_j)^b\varphi$.

In the latter case, we notice that in the logic C^b-S5$_2$, for any formula φ, the following formula is a theorem:

$$(\Box_i\Box_j)^b\varphi \leftrightarrow (\Box_j\Box_i)^b\varphi \tag{3}$$

This immediately follows from axiom $\mathbf{C}^b$.

By using Equation 3 we obtain: $(\Box_i\Box_j)^b\Box_i\varphi \equiv (\Box_j\Box_i)^b\Box_i\varphi$. By repeating the manipulation of the former case and subsequently reapplying Equation 3, we get: $(\Box_j\Box_i)^b\Box_i\varphi \equiv (\Box_j\Box_i)^b\varphi \equiv (\Box_i\Box_j)^b\varphi$. Thus, for each $k \in \mathcal{AG}$, we obtain that the following is a theorem:

$$(\Box_i\Box_j)^b\varphi \rightarrow \Box_{v_1} \cdots \Box_{v_\lambda}\Box_k\varphi.$$

This is the required result. □

The following is the corresponding version of Theorem 2 of Section 4.

Theorem B.1. *Let $i, j \in \mathcal{AG}$ with $i \neq j$ and let φ be any formula. Then, for any b>1, in the logic C^b-S5$_2$ the formula $(\Box_i\Box_j)^b\varphi \leftrightarrow C_{\mathcal{AG}}\varphi$ is a theorem.*

Proof. ($\Leftarrow$) This follows by definition of common knowledge; ($\Rightarrow$) this immediately follows by Lemma B.1. □

The following is the corresponding version of Corollary 1 of Section 4.

Corollary B.1. *Let $i, j \in \mathcal{AG}$ with $i \neq j$, let $\vec{v} \in \mathcal{AG}^*$ and let φ be any formula. Then, for any b>1, in an C^b-S5$_2$-model we have that if $wR_{v_1} \circ \ldots \circ R_{v_{|\vec{v}|}} w'$, then $w(R_i \circ R_j)^b w'$.*

The statement above directly follows from the contrapositive of the implication in Lemma B.1, under the assumption of minimality of models (w.r.t. bisimulation). Intuitively, this states that if a world of a C^b-S5$_2$-model is reachable in an arbitrary number of steps, then it is also reachable in exactly $2b$ steps.

The following is the corresponding version of Lemma 3 of Section 4.

Lemma B.2. *Let (M, W_d) be an C^b-S5$_2$-state, with $M = (W, R, V)$. For any $w, v \in W$, we have that $w \underline{\leftrightarrow}_{2b+1} v \iff w \underline{\leftrightarrow} v$.*

Proof. The proof is identical to that of Lemma 3, by using Corollary B.1 instead of Corollary 1. □

The following is the corresponding version of Lemma 4 of Section 4.

Lemma B.3. *Let (M, W_d) be an C^b-S5$_2$-state, with $M = (W, R, V)$. Then, $|W|$ is bounded in $2b$ and $|\mathcal{P}|$.*

Proof. The proof is identical to that of Lemma 4. □

Theorem 4. *For any $b>1$, PLANEX($\mathcal{T}_{\text{C}^b\text{-S5}}$, 2) is* decidable.

Proof. Let $T \in \mathcal{T}_{\text{C}^b-\text{S5}_2}$ be an epistemic planning task. By Lemma B.3, it follows that we can perform a breadth-first search on the search space that would only visit a finite number of epistemic states (up to bisimulation contraction) to find a solution for T. Thus, we obtain the claim. □

B.1.2 Proof of Theorem 5

To prove Theorem 5, we first show some propaedeutical results (Lemmata B.4, B.5, B.6, B.7).

In what follows, we consider the case with $\mathcal{AG} = \{0, 1, 2\}$ and $b = 2$, *i.e.*, we focus on the logic C^2-S5$_3$. Since C^2-S5$_3$-models are also C^b-S5$_n$-models for any $n > 3$ and $b > 2$, our results hold for any combination of the values of $n \geq 3$ and $b \geq 2$. Moreover, we fix $\mathcal{P} = \{p_1, p_2, p_3, r\}$.

We adapt the proof in [1, Section 6], of the undecidability of epistemic planning in the logic $S5_n$ $(n > 1)$. It is an elegant reduction from the problem of reachability in Minsky two-counter machines to the problem of epistemic planning. We first, recall the definition.

Definition B.1 (Two-counter machine)**.** *A* two-counter machine M *is a finite sequence of instructions $(I_0, \dots, I_T)$, where each instruction I_t, with $t < T$, is from the set:*

$$\{\text{inc}(i), \text{jump}(j), \text{jzdec}(i, j) \mid i = 0, 1, j \leq T\},$$

and $I_T =$ halt*. A* configuration *of M is a triple $(k, l, m) \in \mathbb{N}^3$, where k is the index of the current instruction, and l and m are the current contents of counters 0 and 1, respectively. The* computation function *$f_M : \mathbb{N} \to \mathbb{N}^3$ of M maps time steps into configurations, ad is given by $f_M(0) = (0, 0, 0)$ and if $f_M(n) = (k, l, m)$, then:*

$$f_M(n+1) = \begin{cases} (k+1, l+1, m) & \text{if } I_k = \text{inc}(0) \\ (k+1, l, m+1) & \text{if } I_k = \text{inc}(1) \\ (j, l, m) & \text{if } I_k = \text{jump}(j) \\ (j, l, m) & \text{if } I_k = \text{jzdec}(0, j) \text{ and } l = 0 \\ (j, l, m) & \text{if } I_k = \text{jzdec}(1, j) \text{ and } m = 0 \\ (k+1, l-1, m) & \text{if } I_k = \text{jzdec}(0, j) \text{ and } l > 0 \\ (k+1, l, m-1) & \text{if } I_k = \text{jzdec}(1, j) \text{ and } m > 0 \\ (k, l, m) & \text{if } I_k = \text{halt} \end{cases}$$

We say that M halts *if $f_M(n) = (T, l, m)$ for some $n, l, m \in \mathbb{N}$.*

Theorem B.2 ([5])**.** *The halting problem for two-counter machines is undecidable.*

We follow the approach of [1] step by step by encoding the halting problem of a Minsky machine M as an epistemic planning task. The procedure follows three steps:

1. We define an encoding for integers and configurations;
2. We build a finite set of actions for encoding the computation function f_M; and
3. We combine the previous steps and we encode the halting problem as an epistemic planning task.

In all figures, reflexive, transitive (and symmetric) edges are implicit. In the models, the worlds are labelled with the name of the world and the propositions true in it. In the event models, the events are labelled with the name of the event and the precondition; there are no postconditions.

Integers and configurations. For each $p \in \mathcal{P}$ and each $n \in \mathbb{N}$, we define an epistemic model META-CHAIN(p, n), represented in Figure 2, which contains $n + 1$ meta-worlds (models themselves, Figure 1). Thus, the integer 0 is represented by the meta-chain made of only the meta-world model of Figure 3. Finally, for each configuration $(k, l, m) \in \mathbb{N}^3$ of two-counter machines, we define the epistemic model META-S$_{(k,l,m)}$ as in Figure 4.

In [1], the meta-worlds that compose a meta-chain are always linked together with the same accessibility relation. As a consequence, a meta-chain can have an arbitrary long series of alternating distinct worlds $u_1 \xrightarrow{i} u_2 \xrightarrow{j} u_3 \xrightarrow{i} u_4 \xrightarrow{j} u_5 \cdots$ with i and j distinct and all u_k distinct. This is not possible in the logic C^2-S5$_n$, due to axiom **C**^{2}. Thus, we need to 'bypass' axiom **C**^{2} in the meta-chains. To do so, we devised meta-chains so that meta-worlds are alternatingly linked together with two different relations. This difference also forces us to use three agents instead of two (like in [1]) and, in fact, the plan existence problem in the two agents case is decidable (see Theorem 4).

Notice how in a meta-state, for $i \neq j$, the longest series of alternating distinct worlds $u_1 \xrightarrow{i} u_2 \xrightarrow{j} u_3 \xrightarrow{i} u_4 \xrightarrow{j} u_5 \cdots$ with all u_k distinct, is bounded, and is 4. Thus, the models are thus vacuously models of C^2-S5$_3$.

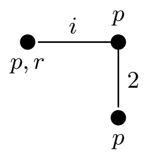

Figure 1: META-WORLD$_i(p)$, with $0 \leq i \leq 1$.

Computation function. First, we define path formulae.

Definition B.2 (Path formulae)**.** *For all $p \in \mathcal{P}$ and $n \in \mathbb{N}$, we inductively define the formulae $\lambda_0(p), \mu_0(p), \tau_0(p)$ as follows:*

- $\lambda_0(p) = p \land \Box_0 \neg r \land \Box_1 \neg r$
- $\mu_0(p) = p \land \Diamond_2 \lambda_0(p) \land \neg \lambda_0(p)$
- $\tau_0(p) = p \land r \land (\Diamond_0 \mu_0(p) \lor \Diamond_1 \mu_0(p))$
- $\lambda_{n+1}(p) = p \land \neg r \land \neg \mu_n(p) \land (\Diamond_0 \mu_n(p) \lor \Diamond_1 \mu_n(p))$
- $\mu_{n+1}(p) = p \land \Diamond_2 \lambda_{n+1}(p) \land \neg \lambda_{n+1}(p)$
- $\tau_{n+1}(p) = p \land r \land (\Diamond_0 \mu_{n+1}(p) \lor \Diamond_1 \mu_{n+1}(p))$

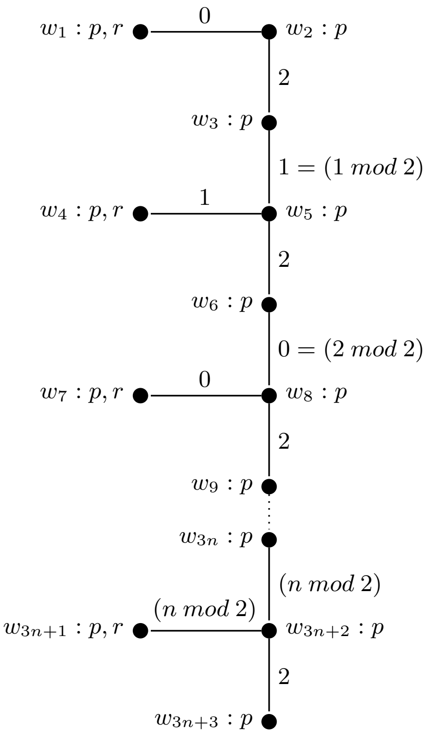

Figure 2: META-CHAIN(p, n): Meta-chains are chains of alternating meta-worlds linked alternatingly by a relation 1 and a relation 0. META-CHAIN(p, n) is a chain of $n + 1$ META-WORLD(p).

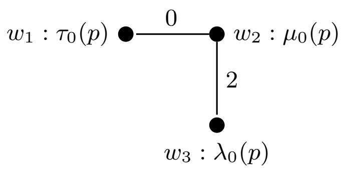

Figure 3: META-CHAIN$(p, 0)$

Lemma B.4. *For all* $p \in \mathcal{P}$, $n \in \mathbb{N}$, $0 \leq i \leq n$, $1 \leq j \leq 3n+3$*:*

$$(\textit{META-CHAIN}(p, n), w_j) \models \lambda_i(p) \quad \Leftrightarrow \quad j=3(n-i)+3$$
$$(\textit{META-CHAIN}(p, n), w_j) \models \mu_i(p) \quad \Leftrightarrow \quad j=3(n-i)+2$$
$$(\textit{META-CHAIN}(p, n), w_j) \models \tau_i(p) \quad \Leftrightarrow \quad j=3(n-i)+1$$

That is, the path formulas allow one to uniquely identify worlds in a meta-chain. In the $(i + 1)th$ to last meta-world in META-CHAIN(p, n), $\lambda_i(p)$ holds in the bottom world, $\mu_i(p)$ in the top-right world, $\tau_i(p)$ in the top-left world. Figure 3 illustrates the base cases.

The instructions of a two-counter machine can be decomposed in simple operations on integers: *increment*, *decrement* and *replacement*. We encode each operation with an event model, represented on Figures 5, 6 and 7, respectively. Due to the structure of meta-chains, that comprise alternating accessibility relations, we need to define two different event models for increment. Namely, META-INC$_0(p)$ is used to increment odd numbers and META-INC$_1(p)$ handles even numbers. Thus, given an integer n, to increment META-CHAIN(p, n), we use META-INC$_i(p)$, where $i = 1 - (n\ mod\ 2)$.

The following Lemma makes sure that the operations on integers are correctly captured by the product update of meta-chains with the event models for increment, decrement and replacement.

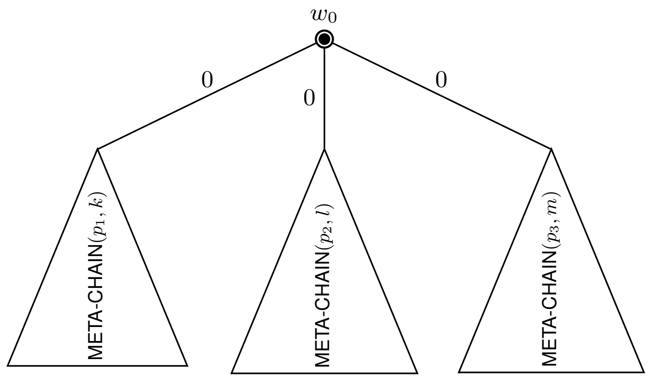

Figure 4: META-S$_{(k,l,m)}$

Lemma B.5. *For all* $m, n \in \mathbb{N}$ *and for all* $p \in \mathcal{P}$*, we have:*

1. *META-CHAIN*$(p, n) \otimes$ *META-INC*$_i(p) =$ *META-CHAIN*$(p, n + 1)$*, where* $i = 1 - (n\ mod\ 2)$*;*
2. *If* $n > 0$*, META-CHAIN*$(p, n) \otimes$ *META-DEC*$(p) =$ *META-CHAIN*$(p, n - 1)$*;*
3. *META-CHAIN*$(p, n) \otimes$ *META-REPL*$(p, n, m) =$ *META-CHAIN*(p, m).

Proof. First, we consider item 1, *i.e.*, event model of Figure 5. Let n be even, *i.e.*, $i = 1$ (the case with $i = 0$ is identical). The top event of Figure 5 is paired with all worlds in META-CHAIN(p, n), except for the one where $\lambda_0(p)$ holds. From Lemma B.4, this world in unique and it is the bottom world of META-CHAIN(p, n). Thus, after the product of META-CHAIN(p, n) with the top event, we obtain a copy of the chain, except for its bottom world. Such world is paired with the second-to-top event of Figure 5. At this point, we obtain an exact copy of META-CHAIN(p, n). Finally, the last three events of Figure 5 create an additional meta-world. Since n is even, the last relation linking meta-worlds in the chain is that of agent 0. Then, the event model META-INC$_1(p)$ links the additional meta-world to the bottom of chain with the accessibility relation of agent 1. Thus, we obtain META-CHAIN$(p, n + 1)$.

We now focus on item 2, *i.e.*, event model of Figure 6. By Lemma B.4, its only event is paired with all worlds of META-CHAIN(p, n), except for those in the bottom meta-world. Since $n > 0$, we obtain META-CHAIN$(p, n - 1)$.

Finally, we consider item 3, *i.e.*, event model of Figure 7. By Lemma B.4, the $2 \cdot (m + 1)$ event models on the right hand side of Figure 7 all pair with the top-right world of the top meta-world of META-CHAIN(p, n) and the $m + 1$ events on the left hand side pair with the top left world of the same meta-world. Thus, we create $m+1$ copies of the top meta-world of META-CHAIN(p, n). Then, these copies are alternatingly linked together with the accessibility relations of agents 0 and 1. Thus, we obtain META-CHAIN(p, m). $\square$

For all $k \in \mathbb{N}$, we define $\phi_k = \Diamond_0 \mu_k(p_1)$. By Lemma B.4 and the definition of META-S$_{(k,l,m)}$, we immediately obtain that for all $k, l, m, k' \in \mathbb{N}$ the following holds:

$$\text{META-S}_{(k,l,m)} \models \phi_{k'} \text{ iff } k' = k. \tag{4}$$

Let now $M = (I_0, \ldots, I_T)$ be a two-counter machine. For all $k < T$ and $l, m \in \mathbb{N}$, we define an epistemic action $a_M(k, l, m)$ as in Figures 8-11.

We now define a notion of *equivalence* between configurations. Two configurations $(k, l, m), (k', l', m') \in \mathbb{N}^3$ are equivalent, de-

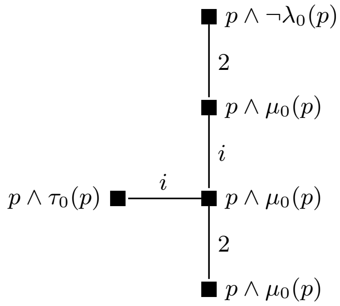

Figure 5: META-INC$_i(p)$, with $0 \leq i \leq 1$.

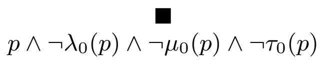

Figure 6: META-DEC(p)

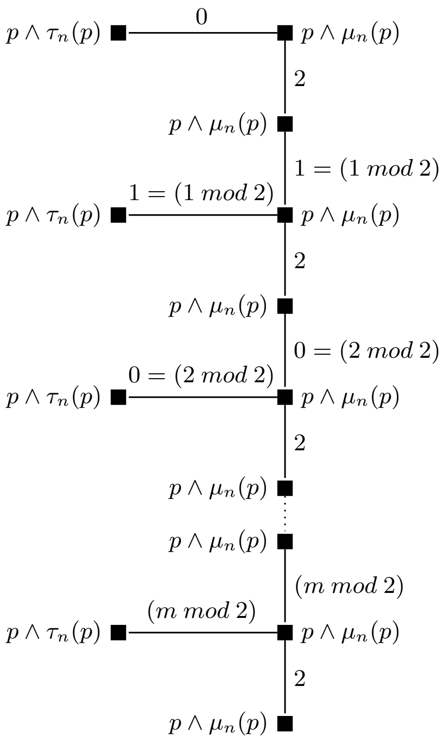

Figure 7: META-REPL(p, n, m)

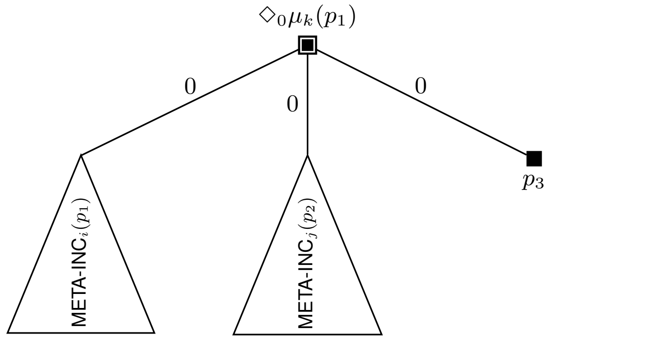

Figure 8: The action $a_M(k, l, m)$ when $I_k = \text{inc}(0)$, $i = 1 - (k \; mod \; 2)$ and $j = 1 - (l \; mod \; 2)$. The case $I_k = \text{inc}(1)$ is obtained by replacing p_2 and p_3 everywhere and by having $j = 1 - (m \; mod \; 2)$.

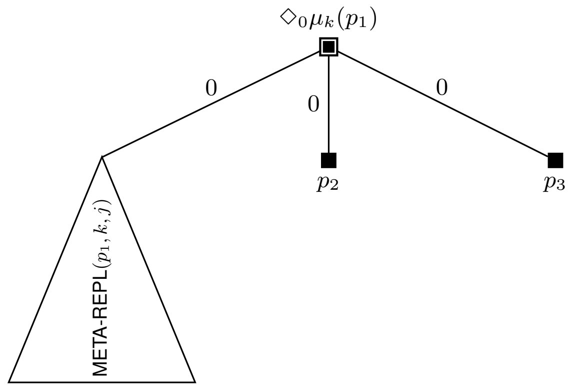

Figure 9: The action $a_M(k, l, m)$ when $I_k = \text{jump}(j)$.

noted by $(k, l, m) \approx (k', l', m')$ if the following holds:

$$k = k' \text{ and } \begin{cases} l = 0 \leftrightarrow l' = 0 & \text{if } I_k = \text{jzdec}(0, j) \\ m = 0 \leftrightarrow m' = 0 & \text{if } I_k = \text{jzdec}(1, j) \end{cases}.$$

Notice that if $(k, l, m) \approx (k', l', m')$, then $a_M(k, l, m) = a_M(k', l', m')$. Thus, the following set is *finite*:

$$\mathcal{F}_M := \{a_M(k, l, m) \mid 0 \leq k < T \text{ and } l, m \in \mathbb{N}\}.$$

The following Lemma shows that $\mathcal{F}_M$ correctly encodes the computation function of the two-counter machine M.

Lemma B.6. *Let $M = (I_0, \ldots, I_T)$ be a two-counter machine, $l, m, n \in \mathbb{N}$ and $k < T$. Then, the following holds:*

1. *$a_M(k, l, m)$ is applicable in META-S$_{f_M(n)}$ iff $(k, l, m) \approx f_M(n)$;*
2. *META-S$_{f_M(n)} \otimes a_M(f_M(n)) =$ META-S$_{f_M(n+1)}$.*

Proof. Let $f_M(n) = (k', l', m')$. The first item by case of I_k.

- $I_k = \text{inc}(0)$, inc(1), or jump(j): $a_M(k, l, m)$ is an action of the form $(\mathcal{E}, \{e\})$ with $pre(e) = \Diamond_0\mu_k(p_1) = \phi_k$. Thus, by equation 4, we have: $a_M(k, l, m)$ is applicable in META-S$_{f_M(n)}$ $\Leftrightarrow$ META-S$_{(k', l', m')} \models \phi_k \Leftrightarrow k = k' \Leftrightarrow (k, l, m) \approx (k', l', m')$.
- $I_k = \text{jzdec}(0, j)$ and $l = 0$: $a_M(k, l, m)$ is an action of the form $(\mathcal{E}, \{e\})$ with $pre(e) = \Diamond_0\mu_k(p_1) \wedge \Diamond_0\mu_0(p_2) = \phi_k \wedge \Diamond_0\mu_0(p_2)$. Thus, by equation 4, we have: $a_M(k, l, m)$ is applicable in META-S$_{f_M(n)}$ $\Leftrightarrow$ META-S$_{(k', l', m')} \models \phi_k \wedge \Diamond_0\mu_0(p_2)$ $\Leftrightarrow k = k' \wedge l' = 0 \Leftrightarrow (k, l, m) \approx (k', l', m')$.

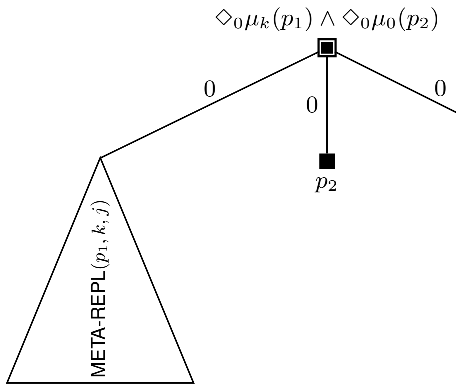

Figure 10: The action $a_M(k,l,m)$ when I_k = jzdec$(0,j), l = 0$. The case I_k = jzdec$(1,j), m = 0$ is obtained by replacing p_2 and p_3 in the precondition of the designated event.

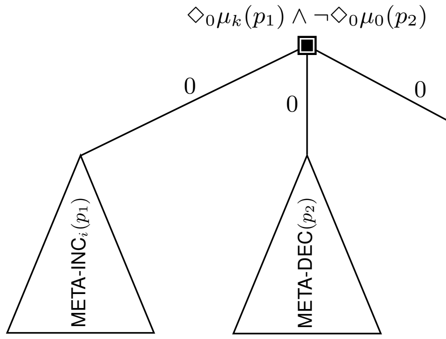

Figure 11: The action $a_M(k,l,m)$ when I_k = jzdec$(0,j), l > 0$ and $i = 1 - (k \ mod \ 2)$. Case I_k = jzdec$(1,j)$ and $m > 0$ is obtained by replacing p_2 and p_3 everywhere.

- I_k = jzdec$(1,j)$ and $m = 0$: analogous to the previous case.
- I_k = jzdec$(0,j)$ and $l > 0$: $a_M(k,l,m)$ is an action of the form $(\mathcal{E}, \{e\})$ with $pre(e) = \Diamond_0\mu_k(p_1) \wedge \neg\Diamond_0\mu_0(p_2) = \phi_k \wedge \neg\Diamond_0\mu_0(p_2)$. Thus, by equation 4, we have: $a_M(k,l,m)$ is applicable in META-S$_{f_M(n)}$ $\Leftrightarrow$ META-S$_{(k',l',m')} \models \phi_k \wedge \neg\Diamond_0\mu_0(p_2) \Leftrightarrow k = k' \wedge l' \neq 0 \Leftrightarrow (k,l,m) \approx (k',l',m')$.
- I_k = jzdec$(1,j)$ and $m > 0$: analogous to the previous case.

The second item is by case of $I_{k'}$:

- $I_{k'}$ = inc(0): $a_M(k',l',m')$ is the action of Figure 8. Thus, by Lemma B.5, we have that: META-S$_{f_M(n)} \otimes a_M(f_M(n))$ = META-S$_{(k',l',m')} \otimes a_M(k',l',m')$ = META-S$_{(k'+1,l'+1,m')}$ = META-S$_{f_M(n+1)}$.
- $I_{k'}$ = inc(1): analogous to the previous case.
- $I_{k'}$ = jump(j): $a_M(k',l',m')$ is the action of Figure 9. Thus, by Lemma B.5, we have that: META-S$_{f_M(n)} \otimes a_M(f_M(n))$ = META-S$_{(k',l',m')} \otimes a_M(k',l',m')$ = META-S$_{(j,l',m')}$ = META-S$_{f_M(n+1)}$.
- $I_{k'}$ = jzdec$(0,j)$ and $l = 0$: $a_M(k',l',m')$ is the action of Figure 10. Thus, by Lemma B.5, we have that: META-S$_{f_M(n)} \otimes a_M(f_M(n))$ = META-S$_{(k',l',m')} \otimes a_M(k',l',m')$ = META-S$_{(j,l',m')}$ = META-S$_{f_M(n+1)}$.
- $I_{k'}$ = jzdec$(1,j)$ and $m = 0$: analogous to the previous case.
- $I_{k'}$ = jzdec$(0,j)$ and $l > 0$: $a_M(k',l',m')$ is the action of Figure 11. Thus, by Lemma B.5, we have that: META-S$_{f_M(n)} \otimes a_M(f_M(n))$ = META-S$_{(k',l',m')} \otimes a_M(k',l',m')$ = META-S$_{(k'+1,l'-1,m')}$ = META-S$_{f_M(n+1)}$.
- $I_{k'}$ = jzdec$(1,j)$ and $m > 0$: analogous to the previous case.

$\square$

Halting problem. From Lemma B.6, we obtain the following result:

Lemma B.7. *Let $M{=}(I_0, \ldots, I_T)$ be a two-counter machine. We define the epistemic planning task T_M = (META-S$_{(0,0,0)}$, $\mathcal{F}_M$, ϕ_T). Then, T_M has a solution iff M halts.*

Thus, from Lemma B.7 and Theorem B.2 and from the fact that C^2-S5$_3$-models are also C^b-S5$_n$-models for any $n > 3$ and $b > 2$, we obtain:

Theorem 5. *For any $n{>}2$ and $b{>}1$,* PLANEX*($\mathcal{T}_{\text{C}^b\text{-S5}}$, n) is* undecidable.

We summarize the results of this section in Table 1.

$\downarrow n$ / $b \rightarrow$	1	2	3	...
1	D	D	D	D
2	D	D	D	D
3	D	UD	UD	UD
4	D	UD	UD	UD
...	D	UD	UD	UD

Table 1: Summary of complexity results for the PLANEX($\mathcal{T}_{\text{C}^b\text{-S5}}$, n). D: decidable; UD: undecidable.

B.2 Weak commutativity

In this section, we give the full proofs of Theorem 6 of Section 4.1.

B.2.1 Proof of Theorem 6

To prove Theorem 6, we first show some propaedeutical results (Lemmata B.8, B.9, Theorem B.3 and Corollary B.2).

The following is the corresponding version of Lemma 1 of Section 4.

Lemma B.8. *Let $G \subseteq \mathcal{AG}$, with $|G| \geq \ell$ and let $\vec{v} \in G^*$ ($|\vec{v}| = \lambda \geq \ell$) such that each agent in G appears in $\vec{v}$ at least once. Let ρ and τ be two permutations of elements of $\vec{v}$. Then, for any φ, in the logic wC$_\ell$-S5$_n$ the following is a theorem:*

$$\Box_{\rho_1} \ldots \Box_{\rho_\lambda}\varphi \leftrightarrow \Box_{\tau_1} \ldots \Box_{\tau_\lambda}\varphi$$

Proof. First, we notice that in the logic wC$_\ell$-S5$_n$, for any formula φ, the following formula is a theorem (recall that $\langle i_1, \ldots, i_\ell \rangle$ is a sequence of agents with no repetitions, and that π is a permutation of this sequence):

$$\Box_{i_1} \ldots \Box_{i_\ell}\varphi \leftrightarrow \Box_{\pi_{i_1}} \ldots \Box_{\pi_{i_\ell}}\varphi \quad (5)$$

This immediately follows from axiom **wC**$_\ell$.

Second, by construction, we have that for each ρ_i there exists τ_{k_i} such that $\tau_{k_i} = \rho_i$. Consider $\tau_{k_1} = \rho_1$. Then, by iterating Equation

5, we obtain:

$$\begin{aligned}
&\Box_{\tau_1}\dots\overbrace{\left(\Box_{\tau_{k_1-1}}\Box_{\tau_{k_1}}\dots\Box_{\tau_{k_1+\ell-1}}\right)}^{\ell}\dots\Box_{\tau_\lambda}\varphi\\
\leftrightarrow\;&\Box_{\tau_1}\dots\left(\Box_{\tau_{k_1}}\Box_{\tau_{k_1-1}}\dots\Box_{\tau_{k_1+\ell-1}}\right)\dots\Box_{\tau_\lambda}\varphi\\
\dots&\\
\leftrightarrow\;&\overbrace{\left(\Box_{\tau_1}\Box_{\tau_{k_1}}\dots\Box_{\tau_j}\right)}^{\ell}\dots\Box_{\tau_{k_1-1}}\dots\Box_{\tau_{k_1+\ell-1}}\dots\Box_{\tau_\lambda}\varphi\\
\leftrightarrow\;&\Box_{\tau_{k_1}}\Box_{\tau_1}\dots\Box_{\tau_j}\dots\Box_{\tau_{k_1-1}}\dots\Box_{\tau_{k_1+\ell-1}}\dots\Box_{\tau_\lambda}\varphi
\end{aligned}$$

By repeating this manipulation for $\rho_2, \dots \rho_m$, we obtain the conclusion. ☐

The following is the corresponding version of Lemma 2 of Section 4.

Lemma B.9. *Let $G = \{i_1, \dots, i_m\} \subseteq \mathcal{AG}$, with $m \geq \ell$. In the logic wC$_\ell$-S5$_n$, for any φ and $\vec{v} \in G^*$ we have that $\Box_{i_1}\dots\Box_{i_m}\varphi \to \Box_{v_1}\cdots\Box_{v_{|\vec{v}|}}\varphi$ is a theorem.*

Proof. The proof is by induction on $|\vec{v}|$. For the base case ($|\vec{v}| = 0$) we have that the formulae $\Box_{i_h}\Box_{i_{h+1}}\dots\Box_{i_m}\varphi \to \Box_{i_{h+1}}\dots\Box_{i_m}\varphi$ ($1 \leq h < m$) and $\Box_{i_m}\varphi \to \varphi$ are instances of **T**. Together with propositional reasoning, we get that $\Box_{i_1}\dots\Box_{i_m}\varphi \to \varphi$ is a theorem.

Let now $|\vec{v}| = \lambda$ and suppose, by inductive hypothesis, that $\Box_{i_1}\dots\Box_{i_m}\varphi \to \Box_{v_1}\cdots\Box_{v_\lambda}\varphi$ is a theorem (for any formula φ). We now show that, for each $j \in G$, the formula $\Box_{i_1}\dots\Box_{i_m}\varphi \to \Box_{v_1}\cdots\Box_{v_\lambda}\Box_j\varphi$ is also a theorem. By inductive hypothesis, substituting φ with $\Box_j\varphi$, the following is a theorem:

$$\Box_{i_1}\dots\Box_{i_m}\Box_j\varphi \to \Box_{v_1}\cdots\Box_{v_\lambda}\Box_j\varphi.$$

Let ρ be any permutation of $G = \{i_1, \dots, i_m\}$, such that $\rho_m = j$. By Lemma B.8, we can now rewrite the formula $\Box_{i_1}\dots\Box_{i_m}\Box_j\varphi$ as $\Box_{\rho_1}\dots\Box_{\rho_m}\Box_j\varphi$, which is $\Box_{\rho_1}\dots\Box_{\rho_m}\Box_{\rho_m}\varphi$. Then, we use Equation 2 as in the proof of Lemma 2 to rewrite the above formula as $\Box_{\rho_1}\dots\Box_{\rho_m}\varphi$. By using Lemma B.8, we can rewrite this formula as $\Box_{i_1}\dots\Box_{i_m}\varphi$. Finally, we obtain that the following is a theorem:

$$\Box_{i_1}\dots\Box_{i_m}\varphi \to \Box_{v_1}\cdots\Box_{v_\lambda}\Box_j\varphi.$$

This is the required result. ☐

The following is the corresponding version of Theorem 2 of Section 4.

Theorem B.3. *Let $G = \{i_1, \dots, i_m\} \subseteq \mathcal{AG}$, with $m \geq \ell$. In the logic wC$_\ell$-S5$_n$, for any φ, the formula $\Box_{i_1}\dots\Box_{i_m}\varphi \leftrightarrow C_G\varphi$ is a theorem.*

Proof. ($\Leftarrow$) This follows by definition of common knowledge; ($\Rightarrow$) this immediately follows by Lemma B.9. ☐

The following is the corresponding version of Corollary 1 of Section 4.

Corollary B.2. *Let $G{=}\{i_1, \dots, i_m\} \subseteq \mathcal{AG}$, with $m \geq \ell$. In an wC$_\ell$-S5$_n$-model, for any $\vec{v} \in G^*$, we have that if $wR_{v_1} \circ \dots \circ R_{v_{|\vec{v}|}}w'$, then $wR_{i_1} \circ \cdots \circ R_{i_m}w'$.*

Figure 12: Frames of S5$_n$-$m\mathcal{A}^*$ event models for ontic actions (left) and sensing/announcement actions (right).

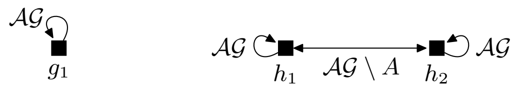

Figure 13: Frames of event models in [4] for *do* and *update* actions (left) and *sense* (right).

The statement above directly follows from the contrapositive of the implication in Lemma B.9, under the assumption of minimality of states (w.r.t. bisimulation). Intuitively, this states that in a wC$_\ell$-S5$_n$-model, given any subset of $m \geq \ell$ agents, if a world is reachable in an arbitrary number of steps, then it is also reachable in exactly m steps. Thus, in general, any pair of worlds of a wC$_\ell$-S5$_n$-model that are reachable one another are connected by a path of length *at most* n.

Theorem 6. *For any $n{>}1$ and $1{<}\ell{\leq}n$,* PlanEx*($\mathcal{T}_{\text{wC}_\ell\text{-S5}}$, n) is decidable.*

Proof. As a result of Corollary B.2, Lemmata 3 and 4 hold also in the logic wC$_\ell$-S5$_n$ (for any $\ell > 1$). Thus, as in the proof of Theorem 3, let $T \in \mathcal{T}_{\text{wC}_\ell\text{-S5}_n}$ be an epistemic planning task (for any such ℓ). By Lemma 4, it follows that we can perform a breadth-first search on the search space that would only visit a finite number of epistemic states (up to bisimulation contraction) to find a solution for T. Thus, we obtain the claim. ☐

C Epistemic Planning Systems

First, we recall that axiom **C** is a Sahlqvist formula that corresponds to the following frame property on event models:

$$\forall u, v, w(uQ_jv \wedge vQ_iw \to \exists x(uQ_ix \wedge xQ_jw)) \tag{6}$$

Lemma 5. *$\mathcal{T}_{\text{S5}_n\text{-}m\mathcal{A}^*} \subseteq \mathcal{T}_{\text{C-S5}}$ and $\mathcal{T}_{\text{KG}} \subseteq \mathcal{T}_{\text{C-S5}}$.*

Proof. We focus on S5$_n$-$m\mathcal{A}^*$ (the proof for the framework by Kominis and Geffner is analogous). It is easy to see that the frames of both public ontic actions (Figure 12, left) and semi-private sensing/announcement actions (Figure 12, right) are reflexive, symmetric and transitive.

We now show that they both satisfy frame property (6). As public ontic actions contain only one event, e_1, this kind of event model trivially satisfies frame property (6). Thus, public ontic actions are C-S5$_n$-actions.

We now move to semi-private sensing/announcement actions. We recall that, by construction in [2], we have that $F \cup P = \mathcal{AG}$ and $F \cap P = \varnothing$. Let $i, j \in \mathcal{AG}$. We now have four cases:

1. $i, j \in F$: we can only assign u, v, w to either $u = v = w = f_1$ or $u = v = w = f_2$. In the former case, frame property (6) is satisfied by choosing $x = f_1$ and, in the latter, by choosing $x = f_2$.
2. $i, j \in P$: for any way of assigning u, v, w, frame property (6) is satisfied by choosing $x = u$.

3. $i \in F$ and $j \in P$: for any way of assigning u, v, w, frame property (6) is satisfied by choosing $x = w$.
4. $i \in P$ and $j \in F$: for any way of assigning u, v, w, frame property (6) is satisfied by choosing $x = u$.

This shows that the event models of semi-private sensing/announcement actions satisfy frame property (6). Thus, semi-private sensing/announcement actions are C-S5_n-actions. This completes the proof. □

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