BETTER BOUNDED BISIMULATION CONTRACTIONS

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Example (Standard k-Contractions are Not Minimal)

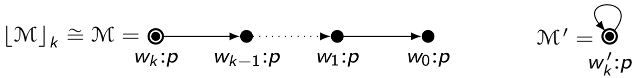


Figure: Chain model $\mathcal{M}$ and standard k-contraction (left) and minimal k-contraction (right).

In this presentation:

- We give an improved definition: **rooted k-contractions**
- We prove correctness and **minimality**
- We show an **exponential succinctness** result

Pointed Models

Let $\mathcal{P}$ be a countable set of atomic propositions and $\mathcal{I}$ a finite set of modality indices.

Definition (Language $\mathcal{L}$ of Multi-Modal Logic)

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_i \varphi, \quad (\text{where } p \in \mathcal{P} \text{ and } i \in \mathcal{I})$$

Definition (Pointed Model)

A **pointed model** is a pair (M, w_d), where w_d is the **designated world** and $M = (W, R, V)$:

- $W \neq \varnothing$ is a finite set of **(possible) worlds**
- $R : \mathcal{I} \to 2^{W \times W}$ assigns to each $i \in \mathcal{I}$ an **accessibility relation** R_i
- $V : \mathcal{P} \to 2^W$ is a **valuation function** assigning to each atom a set of worlds

Bounded Bisimulation

Definition (Bounded Bisimulation)

Let $k \geqslant 0$. A **k-bisimulation** between two pointed models (M, w_d) and (M', w_d'), with $M = (W, R, V)$ and $M' = (W', R', V')$, is a sequence of non-empty binary relations $Z_k \subseteq \cdots \subseteq Z_0 \subseteq W \times W'$ such that $(w_d, w_d') \in Z_k$ and for all $h < k$:

- [atom] If $(w, w') \in Z_0$, then for all $p \in \mathcal{P}$, $w \in V(p)$ iff $w' \in V'(p)$.
- [forth$_h$] If $(w, w') \in Z_{h+1}$ and $wR_i v$, then there is $v' \in W'$ s.t. $w' R_i' v'$ and $(v, v') \in Z_h$.
- [back$_h$] If $(w, w') \in Z_{h+1}$ and $w' R_i' v'$, then there is $v \in W$ s.t. $wR_i v$ and $(v, v') \in Z_h$.

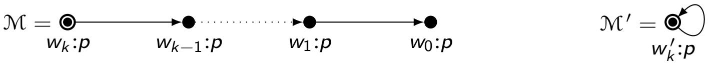
We write $w \leftrightarroweq_k w'$ to denote that w and w' are **k-bisimilar**. We denote the **k-bisimulation (equivalence) class** of $w \in W$ as $[w]_k = \{v \in W \mid w \leftrightarroweq_k v\}$.

Proposition ([book/cup/Blackburn2001])

Two pointed models are k-bisimilar iff they they satisfy the same formulas of $\{\phi \in \mathcal{L} \mid md(\phi) \leqslant k\}$, i.e., *all of formulas up to modal depth k.*

Example

Example (Chain Model)



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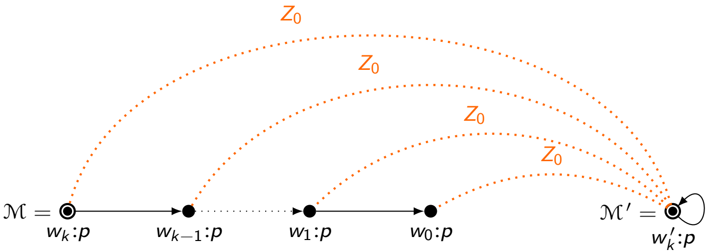

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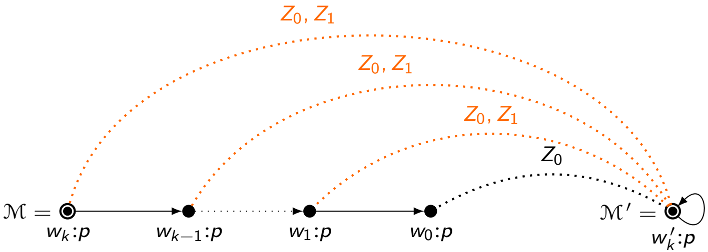

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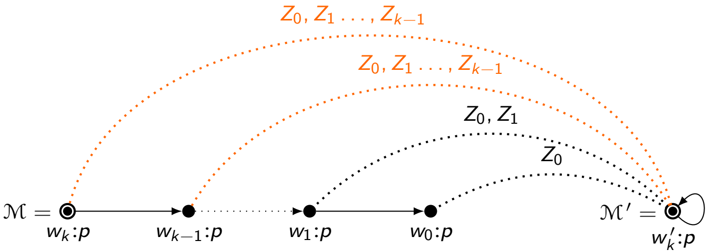

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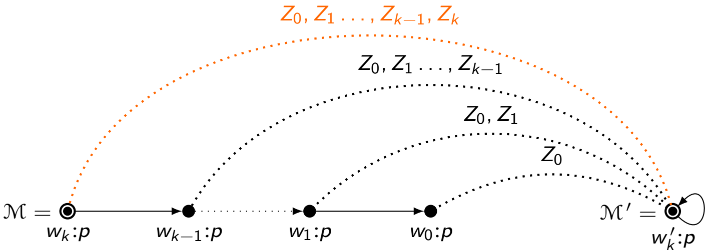

Bisimulation Contractions

Definition (Bisimulation Contraction)

The **(bisimulation) contraction** $\lfloor \mathcal{M} \rfloor$ of a pointed model $\mathcal{M}$ is the **quotient structure** of $\mathcal{M}$ with respect to the bisimulation relation $\leftrightarroweq$. Namely, $\lfloor \mathcal{M} \rfloor = ((W', R', V'), [w_d]_{\leftrightarroweq})$, where:

- $W' = \{[w]_{\leftrightarroweq} \mid w \in W\}$
- $R'_i = \{([w]_{\leftrightarroweq}, [v]_{\leftrightarroweq}) \mid wR_iv\}$
- $V'(p) = \{[w]_{\leftrightarroweq} \in W' \mid w \in V(p)\}$

Proposition ([book/cup/Blackburn2001])

Two pointed models are bisimilar iff they satisfy the same formulas of $\mathcal{L}$.

It is relatively straightforward to prove that:

- $\lfloor \mathcal{M} \rfloor$ is bisimilar to $\mathcal{M}$; and
- $\lfloor \mathcal{M} \rfloor$ is a minimal model bisimilar to $\mathcal{M}$.

Standard k-Contractions

Definition (Standard k-Contraction [journals/logcom/2023/BolanderL; conf/ijcai/Yu2013])

The **standard k-(bisimulation) contraction** $\lfloor \mathcal{M} \rfloor_k$ of a pointed model $\mathcal{M} = ((M, R, V), w_d)$ is the **quotient structure** of $\mathcal{M}$ with respect to $\leftrightarrows_k$. Namely, $\lfloor \mathcal{M} \rfloor_k = ((W', R', V'), [w_d]_k)$:

- $W' = \{[w]_k \mid w \in W\}$
- $R'_i = \{([w]_k, [v]_k) \mid wR_iv\}$
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It is relatively straightforward to prove that $\lfloor \mathcal{M} \rfloor_k$ is k-bisimilar to $\mathcal{M}$.

However. . .

$\lfloor \mathcal{M} \rfloor_k$ is **not** in general a minimal model k-bisimilar to $\mathcal{M}$, as we now show.

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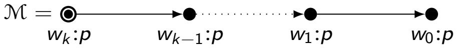

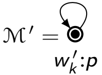

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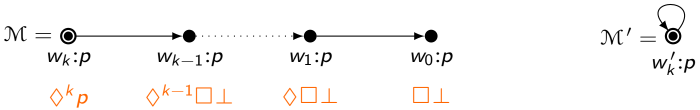

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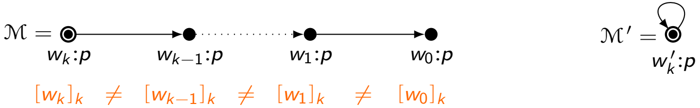

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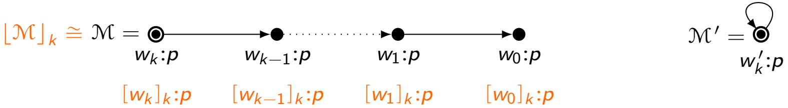

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- Each world of $\mathcal{M}$ can be **identified by a specific formula** of depth $\leqslant k$
- $w_i \neq w_j$ implies $[w_i]_k \neq [w_j]_k$
- $\lfloor \mathcal{M} \rfloor_k$ is **isomorphic** to $\mathcal{M}$

ROOTED K-CONTRACTIONS

Bound of a World

Let $k \geqslant 0$ and let $\mathcal{M} = (M, w_d)$ be a pointed model, with $M = (W, R, V)$.

Definition (Depth and Bound)

- The **depth** $d(w)$ of a world w is the length of the shortest path from w_d to w (∞ if no such path exists).
- The **bound** of a world w is $b(w) = k - d(w)$.

d	b $(k{=}2)$
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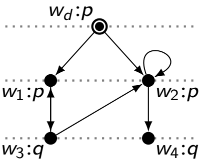
1	1
2	0



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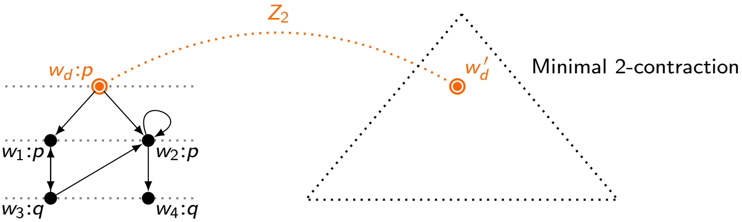

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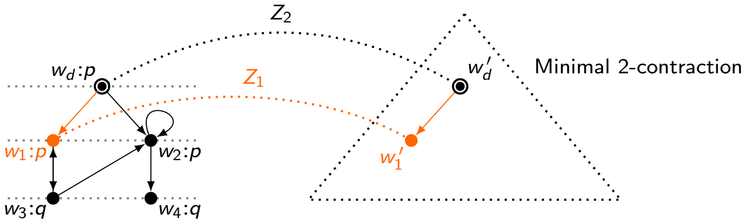

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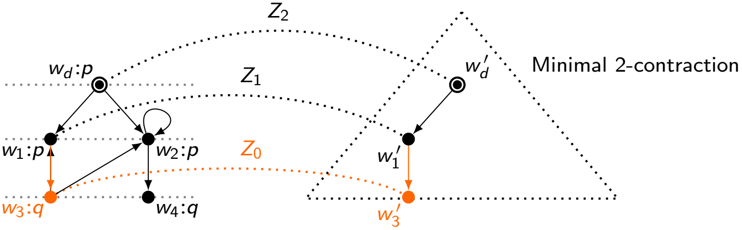

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Redirecting Edges

Example

Figure: Two pointed models: $\mathcal{N}_1$ (left) and $\mathcal{N}_2$ (right).

Let $k = 2$. We have the following:

- $\mathcal{N}_1 \leftrightarroweq_2 \mathcal{N}_2$
- $\mathcal{N}_1$ is **not a world-minimal model** 2-bisimilar to $\mathcal{N}_1$
- $\mathcal{N}_2$ is world minimal, since any model 2-bisimilar to $\mathcal{N}_1$ must have at least three worlds (one for each atomic proposition)

Redirecting Edges

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Figure: Two pointed models: $\mathcal{N}_1$ (left) and $\mathcal{N}_2$ (right).

$\mathcal{N}_2$ is obtained from $\mathcal{N}_1$ by **redirecting** all incoming edges of w_3 to w_2 and deleting the worlds that are no longer reachable from the designated world.

Idea

World w_2 can be considered as a **representative** of w_3.

This is possible because the following two conditions hold:

- $w_2 \leftrightarrows_{b(w_3)} w_3$, namely $w_2 \leftrightarrows_0 w_3$
- $b(w_2) \geqslant b(w_3)$

Example

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Lemma

Let $k \geqslant 0$, let (M, w_d) be a pointed model, with $M = (W, R, V)$, and let $x \in W$ and $y \in W \setminus \{w_d\}$ be two distinct worlds such that:

- $b(x) \geqslant b(y) \geqslant 0$ *and*
- $x \leftrightarroows_{b(y)} y$.

Let (M', w_d) be the pointed model obtained by deleting y from (M, w_d) and **redirecting** *its incoming edges to x. Then, $(M, w_d) \leftrightarroows_k (M', w_d)$.*

Maximal Representatives

Definition (Representatives)

Let x, y be two worlds with non-negative bound.

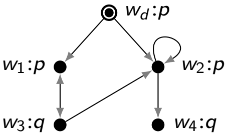
- We say that x **represents** y, denoted by $x \succeq y$, iff $b(x) \geqslant b(y)$ and $x \leftrightarroweq_{b(y)} y$
- If furthermore $b(x) > b(y)$, we say that x **strictly represents** y, denoted by $x \succ y$

Definition (Maximal Representatives)

The set of **maximal representatives** of W is the set of worlds

$$W^{\max} = \{x \in W \mid b(x) \geqslant 0 \text{ and } \neg\exists y \in W(y \succ x)\}$$

Let $k = 1$. **We now calculate** $W^{\max}$:



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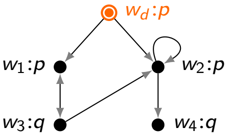

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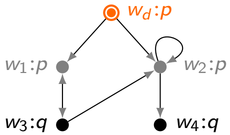

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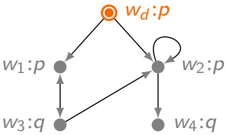

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- $w_d \in W^{\max}$
- $w_1, w_2 \notin W^{\max}$, since $w_d \succ w_1, w_2$
- $w_3, w_4 \notin W^{\max}$, since $b(w_1), b(w_2) < 0$

Thus: $W^{\max} = \{w_d\}$.

Rooted k-Contractions

It is not hard to show that if $w, v \in W^{\max}$ and $w \leftrightarroweq_{b(w)} v$ then $b(w) = b(v)$.

Definition (Representative Class)

The **representative class** of a world w is the class $[w]_{b(w)}$, which we denote with the compact notation $[\![w]\!]$.

Definition (Rooted k-Contraction)

Let $\mathcal{M} = ((W, R, V), w_d)$ and let $k \geqslant 0$. The **rooted k-contraction** of $\mathcal{M}$ is the pointed model $[\![\mathcal{M}]\!]_k = ((W', R', V'), [\![w_d]\!])$, where:

- $W' = \{[\![x]\!] \mid x \in W^{\max}\}$
- $R'_i = \{([\![x]\!], [\![y]\!]) \mid x, y \in W^{\max}, \exists z(xR_iz \text{ and } y \leftrightarroweq_{b(x)-1} z) \text{ and } b(x) > 0\}$
- $V'(p) = \{[\![x]\!] \in W' \mid x \in V(p)\}$

Accessibility relations:

- We **"redirect"** xR_iz to $[\![x]\!]R'_i[\![y]\!]$
- If $b(x) = 0$, then $[\![x]\!]$ only needs to **preserve 0-bisimilarity**

Example

Recall. . .

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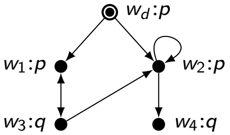

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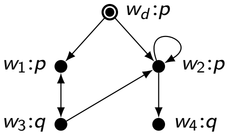

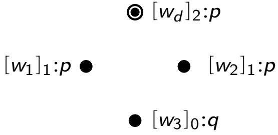

Let $k = 2$:

- $W^{\max} = W$
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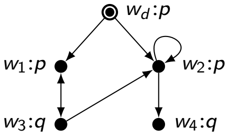

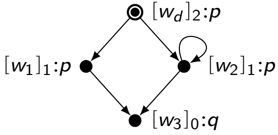

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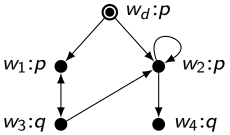

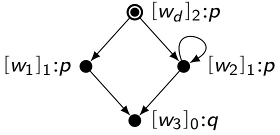

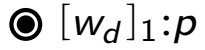

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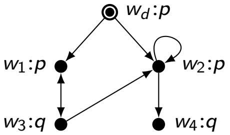
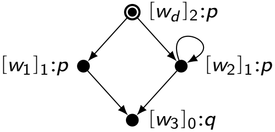
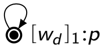
The **rooted k-contraction** of $\mathcal{M}$ is the pointed model $[\![\mathcal{M}]\!]_k = ((W', R', V'), [\![w_d]\!])$, where:

- $W' = \{[\![x]\!] \mid x \in W^{\max}\}$
- $R'_i = \{([\![x]\!], [\![y]\!]) \mid x, y \in W^{\max}, \exists z(xR_i z \text{ and } y \leftrightarrows_{b(x)-1} z) \text{ and } b(x) > 0\}$
- $V'(p) = \{[\![x]\!] \in W' \mid x \in V(p)\}$







Let $k = 1$:

- $W^{\max} = \{w_d\}$
- $[\![w_d]\!] R' [\![w_d]\!]$ since $w_d R w_1$ and $w_d \leftrightarrows_0 w_1$

Correctness

Theorem

Let $\mathcal{M}$ be a pointed model and let $k \geqslant 0$. Then, $\mathcal{M} \leftrightarrows_k \llbracket \mathcal{M} \rrbracket_k$.

Proof idea.

We show that $Z_k, \ldots, Z_0$ is a k-bisimulation between $\mathcal{M}$ and $\llbracket \mathcal{M} \rrbracket_k$, where for all $0 \leqslant h \leqslant k$:

$$Z_h = \{(x, \llbracket x' \rrbracket) \mid x' \in W^{\max}, x' \leftrightarrows_h x \text{ and } b(x) \geqslant h\}.$$

□

World Minimality

Theorem (World Minimality)

Given a pointed model $\mathcal{M}$ *and* $k \geqslant 0$, $\llbracket \mathcal{M} \rrbracket_k$ *is a* **world-minimal model k-bisimilar to $\mathcal{M}$**.

Proof sketch.

Let $\llbracket x \rrbracket \neq \llbracket y \rrbracket$ be two worlds of $\llbracket \mathcal{M} \rrbracket_k$ such that $b(\llbracket x \rrbracket) = b(\llbracket y \rrbracket) = h$:

- We show that $\llbracket x \rrbracket \not\rightleftharpoons_h \llbracket y \rrbracket$
- We then can prove that each set W_h **of worlds of $\llbracket \mathcal{M} \rrbracket_k$ having bound h is minimal**
- If all such sets W_h are minimal, then $\llbracket \mathcal{M} \rrbracket_k$ is **world minimal**

□

MINIMAL K-CONTRACTIONS

We Still Don't Have Edge Minimality

Example

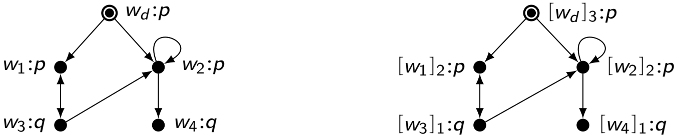


Figure: Pointed models $\mathcal{M}$ (left) and $[\![\mathcal{M}]\!]_3$ (right).

Let $k = 3$. We have $W^{\max} = W$ and $b(w_3) = 1$:

- From $w_3 R w_1$ and $w_3 R w_2$ we have $[\![w_3]\!] R' [\![w_1]\!]$ and $[\![w_3]\!] R' [\![w_2]\!]$
- Since $b(w_3) = 1$ and thus $[\![w_3]\!]$ only needs to preserve 1-bisimilarity to w_3
  - $\rightarrow$ Including only one of those edges in R' is sufficient to guarantee 3-bisimilarity to $\mathcal{M}$

Minimal k-Contractions

Definition

Let $<$ be a total order on W and let $0 \leqslant h \leqslant k$. The **least h-representative** of $w \in W$ is the world $\min_h(w) = \min_<\{v \in W^{\max} \mid v \leftrightarroweq_h w\}$.

Definition (Rooted k-Contractions)

Let $\mathcal{M} = ((W, R, V), w_d)$, let $k \geqslant 0$ and let $<$ be a total order on W. The **rooted k-contraction** of $\mathcal{M}$ wrt. $<$ is the pointed model $\llbracket \mathcal{M} \rrbracket_k^< = ((W', R', V'), \llbracket w_d \rrbracket)$, where:

- $W' = \{\llbracket x \rrbracket \mid x \in W^{\max}\}$;
- $R'_i = \{(\llbracket x \rrbracket, \llbracket \min_{b(x)-1}(y) \rrbracket) \mid x \in W^{\max}, xR_iy \text{ and } b(x) > 0\}$;
- $V'(p) = \{\llbracket x \rrbracket \mid x \in W^{\max} \text{ and } x \in V(p)\}$.

Theorem (Correctness and Minimality)

For any $\mathcal{M}$ and $k \geqslant 0$, $\llbracket \mathcal{M} \rrbracket_k^<$ is a **minimal pointed model k-bisimilar to $\mathcal{M}$**.

EXPONENTIAL SUCCINCTNESS

Exponential Succinctness

Theorem (Exponential succinctness)

There exist models $\mathcal{M}_k$, $k \geqslant 0$, for which the rooted k-contraction has $\Theta(k)$ worlds whereas the standard k-contraction has $\Theta(2^k)$ worlds.

Let $k = 3$. We build $\mathcal{M}_k$ as follows:

$\underline{\varepsilon{:}p_0}$

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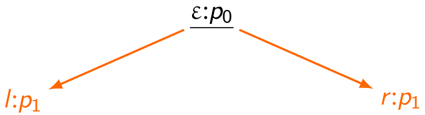

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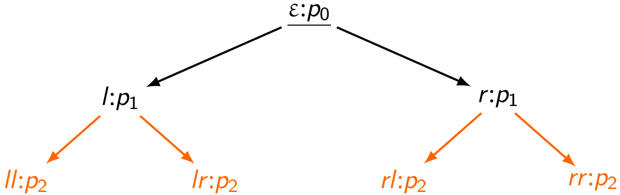

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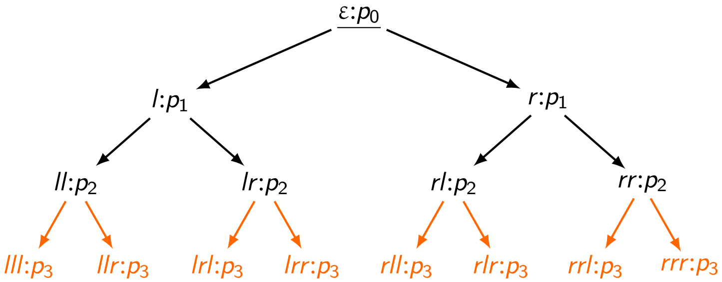

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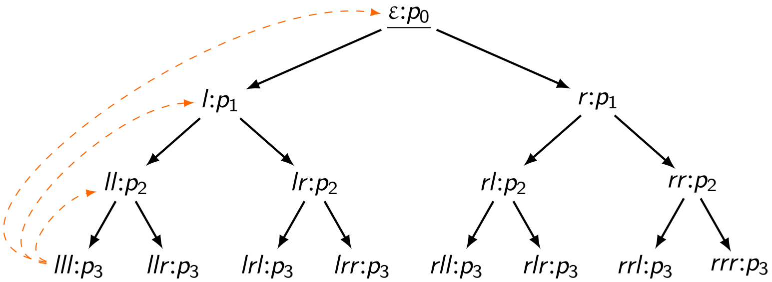

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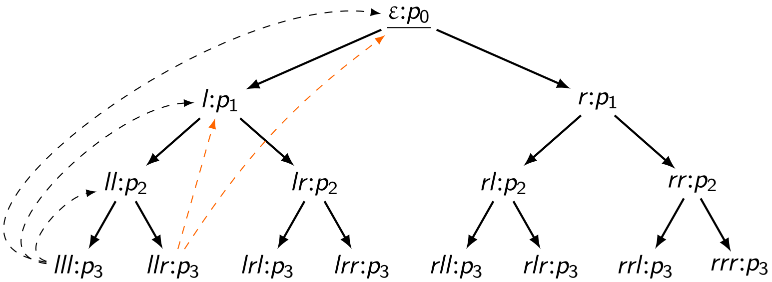

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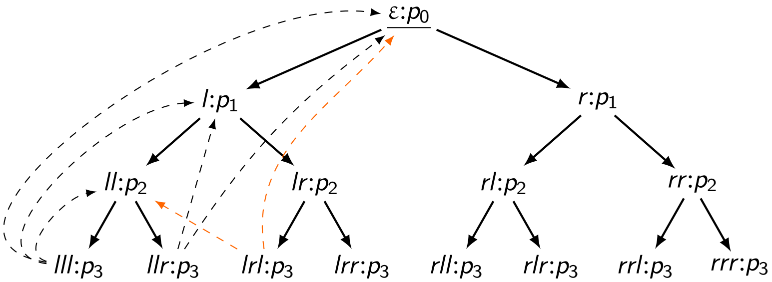

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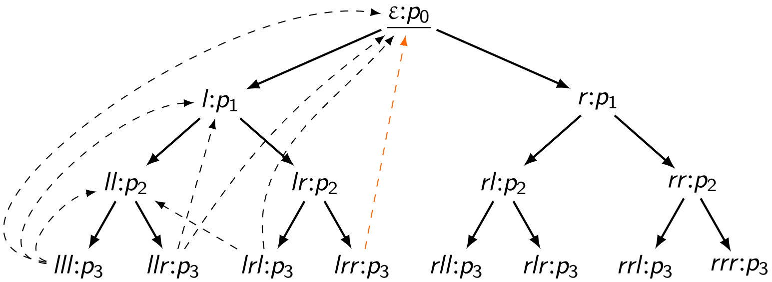

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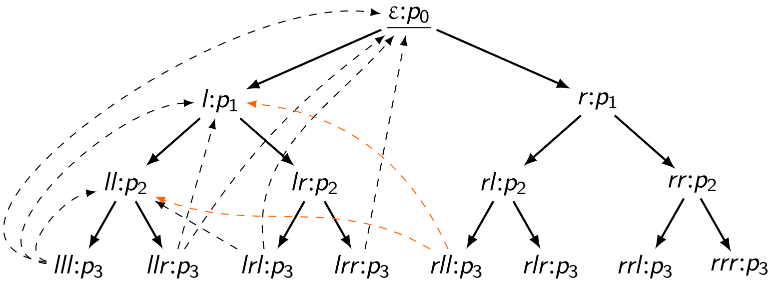

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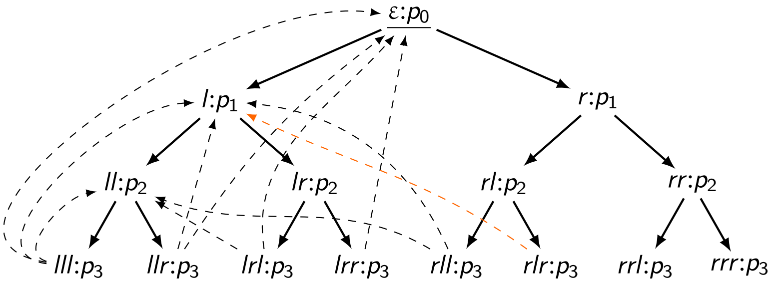

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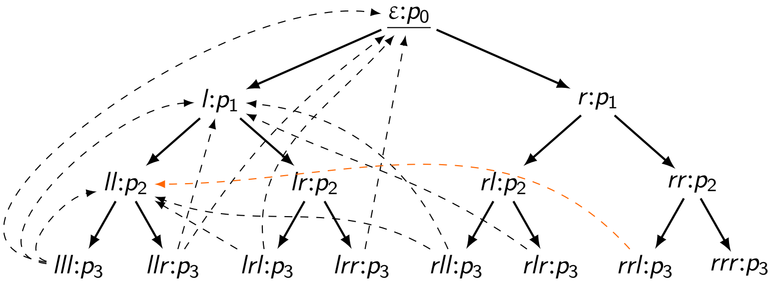

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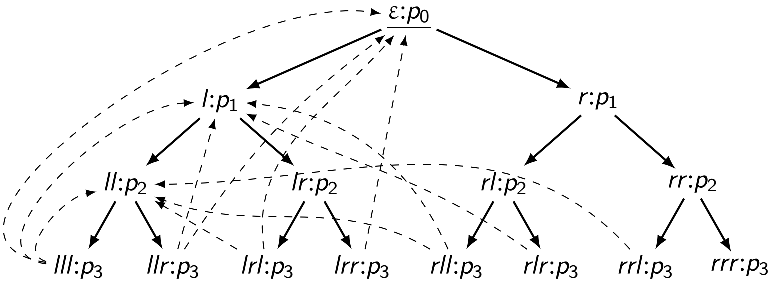

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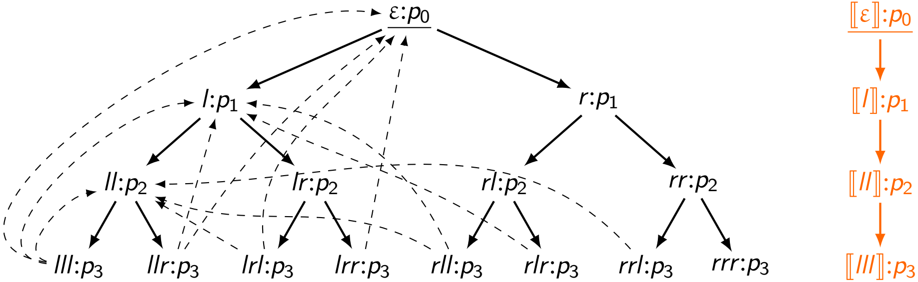

- In $\lfloor \mathcal{M} \rfloor_k$, we have $x \neq y$ **implies** $[x]_k \neq [y]_k$. Thus $|W| = 2^{k+1}$
- In $\lfloor\!\lfloor \mathcal{M} \rfloor\!\rfloor_k^<$, we have $b(x) = b(y)$ **implies** $[\![x]\!] = [\![y]\!]$. Thus $|W| = k + 1$

CONCLUSIONS

Conclusions and Future Works

Rooted k-contractions improve standard k-contractions:

- Provide **minimal** models
- Can be **exponentially more succinct**

Current and future works:

- **Canonical k-contractions**: provide a **unique** minimal k-contracted model
- Multi-pointed models
- Apply rooted k-contractions to **epistemic planning**
- Connection with **knowledge/modal structures** (by R. Fagin, J. Halpern and M. Vardi)

THANK YOU

Questions?